1 Mathematics II

Occasionally the lim will be omitted throughout the script, but the fact that it is a limit will be clear by the comments.

1.1 General Assumptions

- For equations which contain infinity or zero, the conclusion wether the result is zero, infinity or undefined is often hard to make:
  1. $\infty \pm a = \infty \qquad \forall a \in \mathbb{R}$
  2. $\infty + \infty = \infty$
  3. $\infty - \infty$ is undefined
  4. $a \cdot \infty = \begin{cases} \infty & \text{for} \quad a > 0 \\ -\infty & \text{for} \quad a < 0 \\ \text{undefined} & \text{for} \quad a = 0 \end{cases}$
  5. $\frac{\infty}{a} = \begin{cases} \infty & \text{for} \quad a > 0 \\ -\infty & \text{for} \quad a < 0 \end{cases}$
  6. $\frac{a}{\infty} = 0 \qquad \forall a \in \mathbb{R}$
  7. $\infty \cdot \infty = \infty$
  8. $\frac{\infty}{\infty}$ is undefined!
  9. $\frac{a}{0} = \begin{cases} \infty & \text{for} \quad a > 0 \\ -\infty & \text{for} \quad a < 0 \\ undefined & \text{for} \quad a = 0 \end{cases}$
- $f \in C(I)$ means "all functions which are continuous on the interval I.
- $f \in C^n(I)$ means "all functions which are n times continuously on differentiable on the interval I.
- $a^x = e^{x \cdot \ln a}$
- $e = e^1 = exp(1)$
- $\ln e = 1 \qquad \rightsquigarrow \qquad e^x = e^{x \cdot \ln e} = e^{x \cdot 1}$
- The power function follows these rules:

$$\begin{aligned} a^0 &= 1 \\ a^1 &= a \\ a^{x+y} &= a^x \cdot a^y, \quad x, y \in \mathbb{R} \\ (a \cdot b)^x &= a^x \cdot b^x, \quad x \in \mathbb{R} \\ a^{-x} &= \frac{1}{a^x}, \quad x \in \mathbb{R} \\ \log(a^x) &= x \log(a), \quad x \in \mathbb{R} \\ (a^x)^y &= a^{x \cdot y} \neq a^{x^y}, \quad x, y \in \mathbb{R} \end{aligned}$$

- Some rules for trigonometric functions:

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) \\ \cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 \\ \sin(2x) &= 2\sin(x)\cos(x) \end{aligned}$$

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\tan x \cdot \cot x = 1$$
$$1 + \tan^2 x = \frac{1}{\cos^2 x}$$
$$1 + \cot^2 x = \frac{1}{\sin^2 x}$$

1.2 Special Limits

- Some special limits of sequences:

$$\lim_{n\to\infty} \frac{1}{n} = 0$$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$
$$\lim_{n\to\infty} \sqrt[n]{n!} = \infty$$
$$\lim_{n\to\infty} \sqrt[n]{x} = 1, \quad x > 0$$
$$\lim_{n\to\infty} \frac{a^n}{n!} = 0$$
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad \Leftrightarrow \qquad e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{1+n}{n}\right)^n$$
$$\lim_{n\to\infty} k^n = \begin{cases} 0 & \text{for} \quad |k| < 1 \\ 1 & \text{for} \quad k = 1 \end{cases}$$ The sequence (in the last example) diverges for $|k| > 1$

- The EULER transcendent number:

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R}$$

1.3 Sequences

- A sequence is called **arithmetic sequence** if:

$$a_{n+1} - a_n = d = \text{const.} \qquad \text{for} \quad \forall n \in \mathbb{N} \quad \longrightarrow \quad a_n = a_1 + (n-1)d \qquad n \in \mathbb{N}$$

  that is, if the difference between two elements of the sequence is an argument dependent on the first element and the product of n and a constant - hence the difference between two elements is constant.

- A sequence is called **geometric sequence** if:

$$\frac{a_{n+1}}{a_n} = q = \text{const.} \qquad \text{for} \quad \forall n \in \mathbb{N} \quad \longrightarrow \quad a_n = a_1 q^{n-1} \qquad n \in \mathbb{N}$$

  that is, if the difference between two elements of the sequence is an argument determined by a quotient depending on n.

- We say that a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ as n tends to ∞ if, for each $\epsilon > 0$, there is a natural number $N := N(\epsilon)$, such that one has

$$|x_a - a| < \epsilon, \qquad n \geq \mathbb{N}$$

  In this case we write $\lim_{n\to\infty} x_n = a$

- A sequence is called **null sequence** if it converges to zero.
- If a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ then also each of its subsequences converges to a.
- A sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ diverges or is divergent if it is not convergent. In this case we also say that the limit $\lim\limits_{n\to\infty} x_n$ does not exist.
- For $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$ it holds:
  1. $\lim\limits_{n\to\infty} (x_n \pm y_n) = a \pm b$
  2. $\lim\limits_{n\to\infty} (x_n \cdot y_n) = a \cdot b$
  3. $\lim\limits_{n\to\infty} (\alpha x_n) = \alpha \cdot \lim_{n\to\infty} x_n = \alpha \cdot a, \qquad \alpha \in \mathbb{R}$
  4. $\lim\limits_{n\to\infty} (\alpha x_n) = \alpha \cdot \lim_{n\to\infty} x_n = \alpha \cdot a, \qquad \alpha \in \mathbb{R}$
- A sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ is being called **limited** for $\forall n \in \mathbb{N}$, if:
$$|a_n| \leq k \qquad \text{for} \quad k \in \mathbb{R}$$
- A sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ is called **strictly monotonic** if for all $\forall n \in \mathbb{N}$ holds:
$$\begin{aligned} a_{n+1} &> a_n && \text{(strictly monotonic increasing)} \\ a_{n+1} &\geq a_n && \text{(monotonic non-decreasing)} \\ a_{n+1} &< a_n && \text{(strictly monotonic decreasing)} \\ a_{n+1} &\leq a_n && \text{(monotonic non-increasing)} \end{aligned}$$
- If one has to calculate the limit of a sequence, the limit is the quotient of the highest powers of the enumerator n, for the form:
$$a_n = \frac{b \cdot n^w + c \cdot n^x}{d \cdot n^y + e \cdot n^z}$$
Simply divide the whole term by the highest power and cancel everything tending to zero. The result will usually be a simple expression.
- If one has a sequence similar to this:
$$\begin{aligned} \lim_{n\to\infty} a_n &= \left(\frac{n}{n+x}\right)^n \to \quad \text{add \& subtract x:} \quad \left(\frac{n+x-x}{n+x}\right)^n = \left(\frac{n+x}{n+x} - \frac{x}{n+x}\right)^n \\ &= \left(1 - \frac{x}{n+x}\right)^n \quad \text{which yields:} \quad \left(1 \pm \frac{x}{n+x}\right)^n = e^{\pm x} \end{aligned}$$
- If one has something similar to (without denominator):
$$a_n = \sqrt{x} - \sqrt{y} = \left(\sqrt{x} - \sqrt{y}\right) \cdot 1 = \left(\sqrt{x} - \sqrt{y}\right) \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$
Divide or multiply with $\sqrt{x} + \sqrt{y}$ to transform the term into the form $(a - b) \cdot (a + b) = a^2 + b^2$

  Example assuming $\lim_{n\to\infty} a_n$:
$$\begin{aligned} a_n &= n\sqrt{n^4 + n} - n\sqrt{n^4 - n} \quad \longrightarrow \quad \text{substitute:} \quad a = \sqrt{n^4 + n} \quad \& \quad b = \sqrt{n^4 - n} \\ n \cdot (a - b) &= n \cdot (a - b) \cdot 1 = n \cdot (a - b) \cdot \frac{a + b}{a + b} = n \cdot \frac{(a - b) \cdot (a + b)}{a + b} = n \cdot \frac{a^2 + b^2}{a + b} \\ &= n \cdot \left(\frac{(n^4 + n) - (n^4 - n)}{\sqrt{n^4 + n} - \sqrt{n^4 - n}}\right) = \frac{2n^2}{\sqrt{n^4 + n} - \sqrt{n^4 - n}} \\ &= \frac{\frac{2n^2}{n^2}}{\sqrt{\frac{n^4}{n^4} + \frac{n}{n^4}} - \sqrt{\frac{n^4}{n^4} - \frac{n}{n^4}}} = \frac{2}{\sqrt{1 + \frac{1}{n^3}} - \sqrt{1 - \frac{1}{n^3}}} = \frac{1}{\sqrt{1}} = 1 \end{aligned}$$

1.4 Series

For a series $S_k = \sum\limits_{k=1}^{\infty} a_k$ one should first examine the sequence a_k!

- Similarly to arithmetic and geometric sequences we also have **harmonic** and **geometric series**:
  - For the **geometric** series it is even possible to determine the limit:

$$\sum_{k=1}^{\infty} x^k = \begin{cases} \frac{1}{1-x}, & |x| < 1 \\ \text{diverges for } |x| \geq 1 \end{cases}$$

    Example:

$$\sum_{k=0}^{\infty} 9 \cdot \left(\frac{1}{10}\right)^k = 9 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k = 9 \cdot \frac{1}{1-\frac{1}{10}} = 9 \cdot \frac{1}{\frac{9}{10}} = 10 \cdot \frac{9}{9} = 10$$

  - For the **harmonic** series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

  - There also exists the special case of the **alternating harmonic** series:

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}, \quad \text{where the Leibniz test is employed to determine a possible convergence.}$$

  - There are also the so called **power series** (the geometric series is a special case of it):

$$\sum_{k=0}^{\infty} g_k(x), \quad x \in I$$

    where the g_k's are functions defined on an interval I.

- A series may converge, if the limit of the respective sequence tends to zero:

$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \quad \Rightarrow \quad \lim_{k\to\infty} a_k = 0$$

  Proof:

$$s_k := \sum_{k=1}^{\infty} a_k \Rightarrow \begin{cases} \lim\limits_{k\to\infty} s_k & = & s \\ \lim\limits_{k\to\infty} s_{k-1} & = & s \end{cases}$$

  Hence: $a_k = s_k - s_{k-1} \quad \rightsquigarrow \quad s - s = 0 \qquad (n \to \infty)$

  This also implies that, if $\lim\limits_{k\to\infty} a_k \neq 0$ or $\lim\limits_{k\to\infty} a_k$ does not exist $\Rightarrow$ $\sum\limits_{k=1}^{\infty} a_k$ **diverges**!

- **Definition**: $\sum\limits_{k=1} \infty a_k$ is **absolutely convergent** if $\sum\limits_{k=1} \infty |a_k|$. Absolute convergence implies normal convergence!

- **Rearrangement theorem**: If a series is absolutely convergent, then each series formed by rearrangement of its terms converges to the same limit.

- The **radius of convergence** is determined by the following rule for $\sum\limits_{k=a}^{b} c_k$:

$$R = \frac{c_k}{c_{k+1}}$$

- ***Convergence criteria*** for series of the type $\sum_{k=1}^{\infty} a_k$ ...

  1. **Comparison test**: Let some $k_0 \in \mathbb{N}$ exist such that $|a_k| \leq b_k, \quad k \geq k_0$. Then one has
     (a) $\sum_{k=1}^{\infty} b_k$ converges $\rightsquigarrow$ $\sum_{k=1}^{\infty} a_k$ converges absolutely.
     (b) $\sum_{k=1}^{\infty} |a_k| = 0 \quad \rightsquigarrow \quad \sum_{k=1}^{\infty} b_k = 0$

  2. **Ratio test**: If $a_k \neq 0$ is true for all $k \geq k_0$ with some $k_0 \in \mathbb{N}$ and if the sequence $\frac{a_{k+1}}{a_k}$ converges properly or improperly then one has:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} \begin{cases} < 1 & \text{The series converges} \\ > 1 & \text{The series diverges} \\ = 1 & \text{Undefined} \end{cases}$$

     Example assuming ($a_k = \frac{x^k}{k!}; \qquad a_{k+1} = \frac{x^{k+1}}{(k+1)!}$):

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{x^{k+1}\cdot k!}{(k+1)!\cdot x^k} = \lim_{k\to\infty} \frac{x\cdot x^k\cdot k!}{k!\cdot(k+1)\cdot x^k} = \lim_{k\to\infty} \frac{x}{k+1} < 1 \quad \longrightarrow \quad \text{series converges}$$

  3. **Root test**: Let the sequence $\left(\sqrt[k]{|a_k|}\right)_{k\geq 1}$ converge properly or improperly, then the following holds true:

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} \begin{cases} < 1, & \text{The series absolutely converges} \\ > 1, & \text{The series diverges} \\ = 1, & \text{Undefined} \end{cases}$$

     Example assuming ($a_k = \frac{x^k}{k!}$):

$$\sqrt[k]{|a_k|} = \sqrt[k]{\frac{x^k}{k!}} = \frac{\sqrt[k]{x^k}}{\sqrt[k]{k!}} = \frac{x}{\sqrt[k]{k!}} = 0 < 1 \quad \longrightarrow \quad \text{series converges}$$

  4. **Leibniz test**: Assume $c_k \geq c_{k+1} \geq 0, \quad k \in \mathbb{N}$ and $\lim_{k\to\infty} c_k = 0$ then the alternating series $\sum_{k=1}^{\infty} (-1)^k c_k$ converges.

1.5 Polynomials

- Polynomials should first be checked for a zero. Afterwards one divides the polynomial $P(x)$ through the term $(x - x_1)$ where x_1 is the first found zero of the polynomial.
  If the polynomial is of second degree after this, the following approach is used:

$$ax^2 + bx + c \quad \rightsquigarrow \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  or the special case where the quadratic argument has no coefficient:

$$x^2 + px + q \quad \rightsquigarrow \quad \frac{-p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

- If the sum of the coefficients is zero, the first zero of the polynomial is $x_1 = 1$.
  For an "ordered" polynomial $P(x) = a_k x^n + a_{k-1}x^{k-1} + ... + a_1 x + a_0$ one has a_k as the first "odd" coefficient and a_{k+1} as the first "even" coefficient. If there are coefficients which are zero, they should be written to not mix up the order of odd and even elements.

- If the sum of the "odd" and the "even" coefficients is equal, another zero of the polynomial is at $x_2 = -1$.

1.6 Limits of Functions

- For limits of functions there hold similar relations true as for limits of sequences:
  1. $\lim\limits_{x\to x_0}\left(f(x)\pm\pm g(x)\right)=\lim\limits_{x\to x_0}f(x)\pm\lim\limits_{x\to x_0}g(x)=a\pm b$
  2. $\lim\limits_{x\to x_0}\left(f(x)\cdot\pm g(x)\right)=\lim\limits_{x\to x_0}f(x)\cdot\lim\limits_{x\to x_0}g(x)=a\cdot b$
  3. $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}g(x)}=\frac{a}{b}$
- If one has a function of the form:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{0}{0}\quad\text{or}\quad\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

  the rule of L'Hospital says that:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

  Hence if one faces such an undetermined form, derive the two functions and check the limit. In case the first (second, ...) derivative does not yield a suitable solution, just derive once more. Cannot be used with forms like $\frac{a}{0}$ or $\frac{a}{\infty}$ because they are determined as shown in the section **General assumptions**.

1.7 The Derivative

- The definition of the derivative is as follows:

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}\quad\rightsquigarrow\quad\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x)$$

  Example $f(x)=3x^2+7x-8$:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x\to\infty}\frac{\left(3(x+\Delta x)^2+7(x+\Delta x)-8\right)-\left(3x^2+7x-8\right)}{\Delta x}\\
&= \lim_{\Delta x\to\infty}\frac{\left(3(x^2+2x\Delta x+\Delta x^2)+7(x+\Delta x)-8\right)-\left(3x^2+7x-8\right)}{\Delta x}\\
&= \lim_{\Delta x\to\infty}\frac{3x^2+6x\Delta x+3\Delta x^2+7x+7\Delta x-8-3x^2-7x+8}{\Delta x}\\
&= \lim_{\Delta x\to\infty}\frac{6x\Delta x+3\Delta x^2+7\Delta x}{\Delta x}=\lim_{\Delta x\to\infty}\frac{\Delta x\,(6x+3\Delta x+7)}{\Delta x}\\
&= \lim_{\Delta x\to\infty}(6x+3\Delta x+7)=6x+7
\end{aligned}$$

- Some basic derivatives (for valid ranges of definition):

$f(x)$	$f'(x)$	$f''(x)$
$a=const.$	0	0
x^n	nx^{n-1}	$n(n-1)x^{n-2}$
$\sin x$	$\cos x$	$-\sin x$
$\cos x$	$-\sin x$	$-\cos x$
$\tan x$	$\frac{1}{\cos^2 x}=1+\tan^2 x$	$2\tan x(1+\tan^2 x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{x}{(1-x^2)\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$-\frac{x}{(1-x^2)\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\frac{-2x}{(1+x^2)^2}$
e^x	e^x	e^x
$e^{c\cdot x}$	$c\cdot e^{c\cdot x}$	$c^2\cdot e^{c\cdot x}$
$\log_a x$	$\frac{1}{x\cdot\ln a}$	$\frac{-1}{x^2\cdot\ln a}$
$\ln x$	$\frac{1}{x}$	$-\frac{1}{x^2}$

- **Mean value theorem of differentiation**: Let $f \in C^1[a,b]$, then for each pair of points $x, y \in [a,b]$with $x < y$ some point $\xi \in (x,y)$ exists such that

$$f(y) - f(x) = f'(\xi)(x - y)$$

1.8 Taylor Polynomial

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^(n)(x_0)}{n!}(x - x_0)^n + R_n(x)$$

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