

1 Mathematics II

Occasionally the \lim will be omitted throughout the script, but the fact that it is a limit will be clear by the comments.

1.1 General Assumptions

- For equations which contain infinity or zero, the conclusion whether the result is zero, infinity or undefined is often hard to make:

1. $\infty \pm a = \infty \quad \forall a \in \mathbb{R}$

2. $\infty + \infty = \infty$

3. $\infty - \infty$ is undefined

4. $a \cdot \infty = \begin{cases} \infty & \text{for } a > 0 \\ -\infty & \text{for } a < 0 \\ \text{undefined} & \text{for } a = 0 \end{cases}$

5. $\frac{\infty}{a} = \begin{cases} \infty & \text{for } a > 0 \\ -\infty & \text{for } a < 0 \end{cases}$

6. $\frac{a}{\infty} = 0 \quad \forall a \in \mathbb{R}$

7. $\infty \cdot \infty = \infty$

8. $\frac{\infty}{\infty}$ is undefined!

9. $\frac{a}{0} = \begin{cases} \infty & \text{for } a > 0 \\ -\infty & \text{for } a < 0 \\ \text{undefined} & \text{for } a = 0 \end{cases}$

- $f \in C(I)$ means "all functions which are continuous on the interval I ."
- $f \in C^n(I)$ means "all functions which are n times continuously on differentiable on the interval I ."
- $a^x = e^{x \cdot \ln a}$
- $e = e^1 = \exp(1)$
- $\ln e = 1 \quad \rightsquigarrow \quad e^x = e^{x \cdot \ln e} = e^{x \cdot 1}$
- The power function follows these rules:

$$\begin{aligned} a^0 &= 1 \\ a^1 &= a \\ a^{x+y} &= a^x \cdot a^y, \quad x, y \in \mathbb{R} \\ (a \cdot b)^x &= a^x \cdot b^x, \quad x \in \mathbb{R} \\ a^{-x} &= \frac{1}{a^x}, \quad x \in \mathbb{R} \\ \log(a^x) &= x \log(a), \quad x \in \mathbb{R} \\ (a^x)^y &= a^{x \cdot y} \neq a^{x^y}, \quad x, y \in \mathbb{R} \end{aligned}$$

- Some rules for trigonometric functions:

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \sin(x \pm y) &= \sin(x) \cos(y) \pm \cos(x) \sin(y) \\ \cos(x \pm y) &= \cos(x) \cos(y) \mp \sin(x) \sin(y) \\ \cos(2x) &= \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 \\ \sin(2x) &= 2 \sin(x) \cos(x) \end{aligned}$$

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} \\ \cot x &= \frac{\cos x}{\sin x} \\ \tan x \cdot \cot x &= 1 \\ 1 + \tan^2 x &= \frac{1}{\cos^2 x} \\ 1 + \cot^2 x &= \frac{1}{\sin^2 x}\end{aligned}$$

1.2 Special Limits

- Some special limits of sequences:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n} &= 0 \\ \lim_{n \rightarrow \infty} \sqrt[n]{n} &= 1 \\ \lim_{n \rightarrow \infty} \sqrt[n]{n!} &= \infty \\ \lim_{n \rightarrow \infty} \sqrt[n]{x} &= 1, \quad x > 0 \\ \lim_{n \rightarrow \infty} \frac{a^n}{n!} &= 0 \\ \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n &= e^x \quad \Leftrightarrow \quad e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \left(\frac{1+n}{n}\right)^n \\ \lim_{n \rightarrow \infty} k^n &= \begin{cases} 0 & \text{for } |k| < 1 \\ 1 & \text{for } k = 1 \end{cases} \quad \text{The sequence (in the last example) diverges for } |k| > 1\end{aligned}$$

- The EULER transcendent number:

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R}$$

1.3 Sequences

- A sequence is called **arithmetic sequence** if:

$$a_{n+1} - a_n = d = \text{const.} \quad \text{for } \forall n \in \mathbb{N} \quad \longrightarrow \quad a_n = a_1 + (n-1)d \quad n \in \mathbb{N}$$

that is, if the difference between two elements of the sequence is an argument dependent on the first element and the product of n and a constant - hence the difference between two elements is constant.

- A sequence is called **geometric sequence** if:

$$\frac{a_{n+1}}{a_n} = q = \text{const.} \quad \text{for } \forall n \in \mathbb{N} \quad \longrightarrow \quad a_n = a_1 q^{n-1} \quad n \in \mathbb{N}$$

that is, if the difference between two elements of the sequence is an argument determined by a quotient depending on n .

- We say that a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to $a \in \mathbb{R}$ as n tends to ∞ if, for each $\epsilon > 0$, there is a natural number $N := N(\epsilon)$, such that one has

$$|x_n - a| < \epsilon, \quad n \geq N$$

In this case we write $\lim_{n \rightarrow \infty} x_n = a$

- A sequence is called **null sequence** if it converges to zero.
- If a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ converges to $a \in \mathbb{R}$ then also each of its subsequences converges to a .
- A sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ diverges or is divergent if it is not convergent. In this case we also say that the limit $\lim_{n \rightarrow \infty} x_n$ does not exist.
- For $\lim_{n \rightarrow \infty} x_n = a$ and $\lim_{n \rightarrow \infty} y_n = b$ it holds:

1. $\lim_{n \rightarrow \infty} (x_n \pm y_n) = a \pm b$
2. $\lim_{n \rightarrow \infty} (x_n \cdot y_n) = a \cdot b$
3. $\lim_{n \rightarrow \infty} (\alpha x_n) = \alpha \cdot \lim_{n \rightarrow \infty} x_n = \alpha \cdot a, \quad \alpha \in \mathbb{R}$
4. $\lim_{n \rightarrow \infty} (\alpha x_n) = \alpha \cdot \lim_{n \rightarrow \infty} x_n = \alpha \cdot a, \quad \alpha \in \mathbb{R}$

- A sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is being called **limited** for $\forall n \in \mathbb{N}$, if:

$$|a_n| \leq k \quad \text{for } k \in \mathbb{R}$$

- A sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ is called **strictly monotonic** if for all $\forall n \in \mathbb{N}$ holds:

$$\begin{aligned} a_{n+1} &> a_n && \text{(strictly monotonic increasing)} \\ a_{n+1} &\geq a_n && \text{(monotonic non-decreasing)} \\ a_{n+1} &< a_n && \text{(strictly monotonic decreasing)} \\ a_{n+1} &\leq a_n && \text{(monotonic non-increasing)} \end{aligned}$$

- If one has to calculate the limit of a sequence, the limit is the quotient of the highest powers of the numerator n , for the form:

$$a_n = \frac{b \cdot n^w + c \cdot n^x}{d \cdot n^y + e \cdot n^z}$$

Simply divide the whole term by the highest power and cancel everything tending to zero. The result will usually be a simple expression.

- If one has a sequence similar to this:

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \left(\frac{n}{n+x} \right)^n \rightarrow \text{add \& subtract } x: \left(\frac{n+x-x}{n+x} \right)^n = \left(\frac{n+x}{n+x} - \frac{x}{n+x} \right)^n \\ &= \left(1 - \frac{x}{n+x} \right)^n \quad \text{which yields: } \left(1 \pm \frac{x}{n+x} \right)^n = e^{\pm x} \end{aligned}$$

- If one has something similar to (without denominator):

$$a_n = \sqrt{x} - \sqrt{y} = (\sqrt{x} - \sqrt{y}) \cdot 1 = (\sqrt{x} - \sqrt{y}) \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

Divide or multiply with $\sqrt{x} + \sqrt{y}$ to transform the term into the form $(a-b) \cdot (a+b) = a^2 - b^2$

Example assuming $\lim_{n \rightarrow \infty} a_n$:

$$\begin{aligned} a_n &= n\sqrt{n^4+n} - n\sqrt{n^4-n} \rightarrow \text{substitute: } a = \sqrt{n^4+n} \quad \& \quad b = \sqrt{n^4-n} \\ n \cdot (a-b) &= n \cdot (a-b) \cdot 1 = n \cdot (a-b) \cdot \frac{a+b}{a+b} = n \cdot \frac{(a-b) \cdot (a+b)}{a+b} = n \cdot \frac{a^2 - b^2}{a+b} \\ &= n \cdot \frac{(n^4+n) - (n^4-n)}{\sqrt{n^4+n} + \sqrt{n^4-n}} = \frac{2n^2}{\sqrt{n^4+n} + \sqrt{n^4-n}} \\ &= \frac{\frac{2n^2}{n^2}}{\sqrt{\frac{n^4}{n^4} + \frac{n}{n^4}} + \sqrt{\frac{n^4}{n^4} - \frac{n}{n^4}}} = \frac{2}{\sqrt{1 + \frac{1}{n^3}} + \sqrt{1 - \frac{1}{n^3}}} = \frac{1}{\sqrt{1}} = 1 \end{aligned}$$

1.4 Series

For a series $S_k = \sum_{k=1}^{\infty} a_k$ one should first examine the sequence a_k !

- Similarly to arithmetic and geometric sequences we also have **harmonic** and **geometric series**:

– For the **geometric** series it is even possible to determine the limit:

$$\sum_{k=1}^{\infty} x^k = \begin{cases} \frac{1}{1-x}, & |x| < 1 \\ \text{diverges for } |x| \geq 1 \end{cases}$$

Example:

$$\sum_{k=0}^{\infty} 9 \cdot \left(\frac{1}{10}\right)^k = 9 \cdot \sum_{k=0}^{\infty} \left(\frac{1}{10}\right)^k = 9 \cdot \frac{1}{1 - \frac{1}{10}} = 9 \cdot \frac{10}{9} = 10$$

– For the **harmonic** series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

– There also exists the special case of the **alternating harmonic** series:

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}, \quad \text{where the LEIBNIZ test is employed to determine a possible convergence.}$$

– There are also the so called **power series** (the geometric series is a special case of it):

$$\sum_{k=0}^{\infty} g_k(x), \quad x \in I$$

where the g_k 's are functions defined on an interval I .

- A series may converge, if the limit of the respective sequence tends to zero:

$$\sum_{k=1}^{\infty} a_k \quad \text{converges} \quad \Rightarrow \quad \lim_{k \rightarrow \infty} a_k = 0$$

Proof:

$$s_k := \sum_{k=1}^{\infty} a_k \Rightarrow \begin{cases} \lim_{k \rightarrow \infty} s_k = s \\ \lim_{k \rightarrow \infty} s_{k-1} = s \end{cases}$$

Hence: $a_k = s_k - s_{k-1} \rightsquigarrow s - s = 0 \quad (n \rightarrow \infty)$

This also implies that, if $\lim_{k \rightarrow \infty} a_k \neq 0$ or $\lim_{k \rightarrow \infty} a_k$ does not exist $\Rightarrow \sum_{k=1}^{\infty} a_k$ **diverges!**

- **Definition:** $\sum_{k=1}^{\infty} \infty a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} \infty |a_k|$. Absolute convergence implies normal convergence!
- **Rearrangement theorem:** If a series is absolutely convergent, then each series formed by rearrangement of its terms converges to the same limit.
- The **radius of convergence** is determined by the following rule for $\sum_{k=a}^b c_k$:

$$R = \frac{c_k}{c_{k+1}}$$

- **Convergence criteria** for series of the type $\sum_{k=1}^{\infty} a_k \dots$

1. **Comparison test:** Let some $k_0 \in \mathbb{N}$ exist such that $|a_k| \leq b_k$, $k \geq k_0$. Then one has

$$\begin{aligned} \text{(a)} \quad \sum_{k=1}^{\infty} b_k \quad \text{converges} &\quad \rightsquigarrow \quad \sum_{k=1}^{\infty} a_k \quad \text{converges absolutely.} \\ \text{(b)} \quad \sum_{k=1}^{\infty} |a_k| = 0 &\quad \rightsquigarrow \quad \sum_{k=1}^{\infty} b_k = 0 \end{aligned}$$

2. **Ratio test:** If $a_k \neq 0$ is true for all $k \geq k_0$ with some $k_0 \in \mathbb{N}$ and if the sequence $\frac{a_{k+1}}{a_k}$ converges properly or improperly then one has:

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} \begin{cases} < 1 & \text{The series converges} \\ > 1 & \text{The series diverges} \\ = 1 & \text{Undefined} \end{cases}$$

Example assuming $(a_k = \frac{x^k}{k!}; \quad a_{k+1} = \frac{x^{k+1}}{(k+1)!})$:

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} = \lim_{k \rightarrow \infty} \frac{x^{k+1} \cdot k!}{(k+1)! \cdot x^k} = \lim_{k \rightarrow \infty} \frac{x \cdot x^k \cdot k!}{k! \cdot (k+1) \cdot x^k} = \lim_{k \rightarrow \infty} \frac{x}{k+1} < 1 \quad \longrightarrow \quad \text{series converges}$$

3. **Root test:** Let the sequence $(\sqrt[k]{|a_k|})_{k \geq 1}$ converge properly or improperly, then the following holds true:

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} \begin{cases} < 1, & \text{The series absolutely converges} \\ > 1, & \text{The series diverges} \\ = 1, & \text{Undefined} \end{cases}$$

Example assuming $(a_k = \frac{x^k}{k!})$:

$$\sqrt[k]{|a_k|} = \sqrt[k]{\frac{x^k}{k!}} = \frac{\sqrt[k]{x^k}}{\sqrt[k]{k!}} = \frac{x}{\sqrt[k]{k!}} = 0 < 1 \quad \longrightarrow \quad \text{series converges}$$

4. **Leibniz test:** Assume $c_k \geq c_{k+1} \geq 0$, $k \in \mathbb{N}$ and $\lim_{k \rightarrow \infty} c_k = 0$ then the alternating series $\sum_{k=1}^{\infty} (-1)^k c_k$ converges.

1.5 Polynomials

- Polynomials should first be checked for a zero. Afterwards one divides the polynomial $P(x)$ through the term $(x - x_1)$ where x_1 is the first found zero of the polynomial.

If the polynomial is of second degree after this, the following approach is used:

$$ax^2 + bx + c \quad \rightsquigarrow \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or the special case where the quadratic argument has no coefficient:

$$x^2 + px + q \quad \rightsquigarrow \quad \frac{-p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

- If the sum of the coefficients is zero, the first zero of the polynomial is $x_1 = 1$.
For an "ordered" polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ one has a_n as the first "odd" coefficient and a_{n+1} as the first "even" coefficient. If there are coefficients which are zero, they should be written to not mix up the order of odd and even elements.
- If the sum of the "odd" and the "even" coefficients is equal, another zero of the polynomial is at $x_2 = -1$.

1.6 Limits of Functions

- For limits of functions there hold similar relations true as for limits of sequences:

- $\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = a \pm b$
- $\lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = a \cdot b$
- $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{a}{b}$

- If one has a function of the form:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

the rule of L'HOSPITAL says that:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$$

Hence if one faces such an undetermined form, derive the two functions and check the limit. In case the first (second, ...) derivative does not yield a suitable solution, just derive once more. Cannot be used with forms like $\frac{a}{0}$ or $\frac{a}{\infty}$ because they are determined as shown in the section **General assumptions**.

1.7 The Derivative

- The definition of the derivative is as follows:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \rightsquigarrow \quad \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

Example $f(x) = 3x^2 + 7x - 8$:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow \infty} \frac{(3(x + \Delta x)^2 + 7(x + \Delta x) - 8) - (3x^2 + 7x - 8)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow \infty} \frac{(3(x^2 + 2x\Delta x + \Delta x^2) + 7(x + \Delta x) - 8) - (3x^2 + 7x - 8)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow \infty} \frac{3x^2 + 6x\Delta x + 3\Delta x^2 + 7x + 7\Delta x - 8 - 3x^2 - 7x + 8}{\Delta x} \\ &= \lim_{\Delta x \rightarrow \infty} \frac{6x\Delta x + 3\Delta x^2 + 7\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow \infty} \frac{\Delta x(6x + 3\Delta x + 7)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow \infty} (6x + 3\Delta x + 7) = 6x + 7 \end{aligned}$$

- Some basic derivatives (for valid ranges of definition):

$f(x)$	$f'(x)$	$f''(x)$
$a = \text{const.}$	0	0
x^n	nx^{n-1}	$n(n-1)x^{n-2}$
$\sin x$	$\cos x$	$-\sin x$
$\cos x$	$-\sin x$	$-\cos x$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$2 \tan x(1 + \tan^2 x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{x}{(1-x^2)\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$-\frac{x}{(1-x^2)\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\frac{-2x}{(1+x^2)^2}$
e^x	e^x	e^x
$e^{c \cdot x}$	$c \cdot e^{c \cdot x}$	$c^2 \cdot e^{c \cdot x}$
$\log_a x$	$\frac{1}{x \cdot \ln a}$	$\frac{-1}{x^2 \cdot \ln a}$
$\ln x$	$\frac{1}{x}$	$-\frac{1}{x^2}$

- **Mean value theorem of differentiation:** Let $f \in C^1[a, b]$, then for each pair of points $x, y \in [a, b]$ with $x < y$ some point $\xi \in (x, y)$ exists such that

$$f(y) - f(x) = f'(\xi)(x - y)$$

1.8 Taylor Polynomial

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

References

- [1] REEMTSEN Rempbert Prof. Dr.
Mathematics II - Calculus.
2004 BTU Cottbus.
- [2] MEYBERG Kurt, VACHENAUER Peter.
Höhere Mathematik 1 und 2.
1991 Springer Verlag Berlin - Heidelberg - New York.
- [3] RAPSCH Ilse.
Folgen - Reihen - Grenzwerte.
1998 Verlag Franzbecker, Hildesheim, Berlin.
- [4] PRECHT Dr. Manfred, VOIT Dipl.-Math. Karl, KRAFT Dipl.-Ing. agr. Roland.
Mathematik 2 für Nichtmathematiker.
1991 R. Oldenbourg Verlag München Wien.
- [5] SCHEID Prof. Dr. Harald (Bearb.), ENGESSER Hermann (Hrsg.).
Duden Rechnen und Mathematik: das Lexikon für Schule und Praxis.
1994 Bibliographisches Institut & F.A. Brockhaus AG, Mannheim.